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Algorithmic Blending

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An algorithm is given by which one can obtain a 4-sided C^1 surface patch. The algorithm can be used to interpolate 4 arbitrary polynomial boundary curves together with corresponding tangent planes. The method presented can be generalized to produce C^k surfaces which are piecewise polynomial of degree $2k+1$ or higher and which interpolate arbitrary C^k boundary information. Furthermore, the method can be used with triangular patches. © 1993 Academic Press, Inc.

1. GENERAL DESCRIPTION

We describe an iterative method by which one can construct a 4-sided C^1 surface patch. This method can be used to interpolate 4 arbitrary polynomial boundary curves together with corresponding tangent planes.

Other solutions to this interpolation problem have been given earlier by Coons [3] and for more general situations by Gregory [5]. Gregory's solution is a rational surface of degree $\frac{5}{2}$ with singularities at its corners. The method presented here produces a simpler patch, namely, a piecewise bicubic polynomial with, in general, infintely many polynomial pieces. The method is a simple averaging as it is derived from the de Casteljau algorithm and the midpoint computation $\frac{1}{2}(a+b)$.

Moreover the method is immediately generalizable to C^k patches which interpolate arbitrary C^k information along their boundaries [6]. Then the patches are piecewise polynomials of degree $2k+1$. For $k=2$ this solution can be compared to the extension of Gregory's patch by Barnhill [1] and Worsey [8] or Takai and Wang [9]. The solutions given there are rational of degree $\frac{11}{7}$ or higher with singularities at their corners.

Also, the method presented below can be modified to produce triangular C^k patches which are piecewise polynomials of degree $2k+1$ and which interpolate arbitrary C^k information along their boundaries [7].

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2. The Set Up

To start with, let $\mathbf{a}_i(u, v)$, $\mathbf{b}_i(u, v)$, $i=0, 1$, be four patches in $C^{1,1}([0\ 1]^2, R^m)$, $m \geqslant 2$. They provide the boundary information for the patch $\mathbf{p}$ which is to be constructed, cf. Fig. 1.

This means the stipulation

$$\mathbf{p}(i, j) = \mathbf{a}_j(i, 1-j) = \mathbf{b}_i(1-i, j), \qquad i, j \in \{0, 1\}, \tag{2.1}$$

and that $\mathbf{a}_j$ and $\mathbf{b}_i$ have a common tangent plane at $\mathbf{p}(i, j)$.

We do not require, however, that

$$\frac{d}{du}\frac{d}{dv}\mathbf{a}_j(i, 1-j) = \frac{d}{du}\frac{d}{dv}\mathbf{b}_i(1-i, j) \tag{2.2}$$

nor do we suppose that

$$\frac{d}{dw}\mathbf{a}_j(i, 1-j) = \frac{d}{dw}\mathbf{b}_i(1-i, j), \qquad w = u, v, \tag{2.3}$$

for any $(i, j) \in \{0, 1\}^2$.

The patch $\mathbf{p}$ will be constructed as a piecewise bicubic such that $\mathbf{p}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, and $\mathbf{b}_1$ together form a continuous and tangent plane continuous surface.

For the construction of $\mathbf{p}$ one needs the following:

1. For any 4×4 matrix $C = [\mathbf{c}_{i,j}]_{i,j=0}^3$ over R^m the corresponding bicubic Bézier patch is denoted by

$$\mathscr{B}[C](u, v) := \begin{cases} \sum \mathbf{c}_{i,j} \binom{3}{i} u^i (1-u)^{3-i} \binom{3}{j} v^j (1-v)^{3-j}, & (u, v) \in [0, 1]^2 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

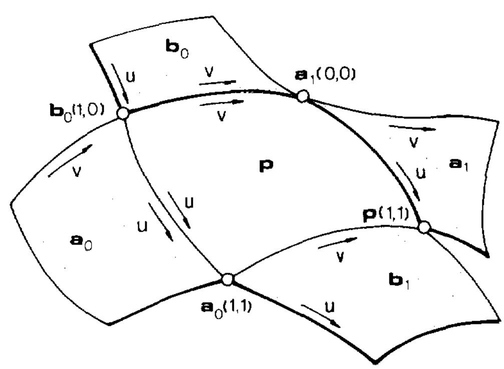

Fig. 1. The patch to be constructed

and for any 8×8 matrix over R^m with 4×4 blocks D_i,

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

the corresponding piecewise bicubic Bézier patch over $[0, 1]^2$ is defined by

$$\begin{aligned} \tilde{\mathscr{B}}[D](u, v) = \mathscr{B}[D_1](2u, 2v) + \mathscr{B}[D_2](2u, 2v-1) \\ + \mathscr{B}[D_3](2u-1, 2v) + \mathscr{B}[D_4](2u-1, 2v-1). \end{aligned} \tag{2.5}$$

(For more information on the Bézier representation of polynomials cf. [2].)

2. Let $A = [\mathbf{a}_{ij}]_{i,j=0}^3$ be a Bézier matrix such that the surface

$$\mathbf{a}(u, v) = \mathscr{B}[A](u, v) \tag{2.6}$$

interpolates $\mathbf{a}_0$ and $\mathbf{a}_1$ at the corners, i.e.,

$$\begin{aligned} \mathbf{a}(i, j) &= \mathbf{a}_j(i, 1-j), \\ \frac{d}{dv}\mathbf{a}(i, j) &= \frac{d}{dv}\mathbf{a}_j(i, 1-j), \\ \frac{d}{du}\mathbf{a}(i, j) &= \frac{d}{du}\mathbf{a}_j(i, 1-j) \end{aligned}$$

and

$$\frac{d^2}{du\,dv}\mathbf{a}(i, j) = \frac{d^2}{du\,dv}\mathbf{a}_j(i, 1-j).$$

Similarly let $B = [\mathbf{b}_{i,j}]_{i,j=0}^3$ be such that

$$\mathbf{b}(u, v) = \mathscr{B}[B](u, v) \tag{2.7}$$

interpolates $\mathbf{b}_0$ and $\mathbf{b}_1$ in a similar fashion, i.e.,

$$D[\mathbf{b}](i, j) = D[\mathbf{b}_i](1-i, j)$$

for

$$D = id, \frac{d}{dv}, \frac{d}{du}, \frac{d^2}{dv\,du}, \quad (i, j) \in \{0, 1\}^2.$$

Note that because of (2.1), $\mathbf{a}_{k,l} = \mathbf{b}_{k,l}$ for all $k, l \in \{0, 3\}$.

3. Both $\mathbf{a}$ and $\mathbf{b}$ will be successively subdivided (here "quartered") and partly averaged after each subdivision. If $C = [\mathbf{c}_{i,j}]_{i,j=0}^3$ is a (Bézier)

matrix, then $\tilde{C} = [\tilde{\mathbf{c}}_{i,j}]_{i,j=0}^{7}$ will denote the (Bézier) matrix obtained from C by subsidivion, i.e., $\tilde{C} = U^{\mathrm{T}} C U$, where

$$U = \frac{1}{8} \begin{bmatrix} 8 & 4 & 2 & 1 & 1 & & & \\ & 4 & 4 & 3 & 3 & 2 & & \\ & & 2 & 3 & 3 & 4 & 4 & \\ & & & 1 & 1 & 2 & 4 & 8 \end{bmatrix}.$$

This means that $\mathscr{B}[C](u, v) = \tilde{\mathscr{B}}[\tilde{C}](u, v)$.

3. The Construction for Cubic Boundaries

The goal is to solve the above interpolation problem by only a piecewise bicubic patch. A further motivation is that the underlying construction for this patch is a very simple modified subdivision method which in contrast to other simple subdivision methods, e.g., [4], allows for an easy analysis of the surface it generates.

The construction of the interpolating patch $\mathbf{p}$ is most simply described if the $\mathbf{a}_i$ and $\mathbf{b}_i$ are bicubic. It is given through the recursive procedure blend below which has two 4×4 matrices as parameters. Modifications of this procedure which work for arbitrary boundaries $\mathbf{a}_i$, $\mathbf{b}_i$ are given in Section 10.

ALGORITHM. blend (A, B)

`begin`
  `if` $A = B$
  `then` render the bicubic patch $\mathscr{B}[A]$
  `else`
  `begin` Form $M := (\tilde{A} + \tilde{B})/2 = [m_{ij}]_{i,j=0}^{7}$
Define the eight 4×4 matrices A_i and B_i, $i = 1, 2, 3, 4$, by

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} + & + & - & - & \vdots & - & - & + & + \\ + & + & - & - & \vdots & - & - & + & + \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ + & + & - & - & \vdots & - & - & + & + \\ + & + & - & - & \vdots & - & - & + & + \end{bmatrix} \tag{3.1}$$

and

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \left[\begin{array}{cccc c cccc} - & - & - & - & \vdots & - & - & - & - \\ - & - & - & - & \vdots & - & - & - & - \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ + & + & \circ & \circ & \vdots & \circ & \circ & + & + \\ - & - & - & - & \vdots & - & - & - & - \\ - & - & - & - & \vdots & - & - & - & - \end{array}\right] \tag{3.2}$$

where

$$\text{entry}(i, j) = \begin{cases} \tilde{a}_{i,j} & \text{if marked by } + \\ \tilde{b}_{i,j} & \text{if marked by } -. \\ m_{i,j} & \text{if marked by } \circ \end{cases}$$

For $i = 1, 2, 3, 4$
execute blend (A_i, B_i)
end of else
end of procedure.

4. First Discussion

Figure 2 shows what blend (A, B) has rendered at the third recursion level if $A = B$ except, e.g., $\mathbf{a}_{2,1} \neq \mathbf{b}_{2,1}$ and $\mathbf{a}_{3,1} \neq \mathbf{b}_{3,1}$. A question mark points out the hole not yet filled. The numbers refer to the recursion level at which the corresponding patch is generated.

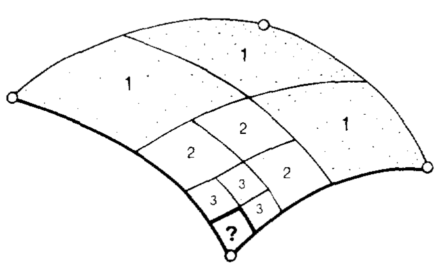

FIG. 2. The patch blend (A, B) after three recursions.

In general, the procedure blend renders successively more and more bicubic patches which fill the "hole" between $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, $\mathbf{b}_1$ in the limit. Together with the boundary curves of $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, $\mathbf{b}_1$ these infinitely many patches form a C^1 surface patch $\mathbf{p}$ which fits C^1 smoothly in the hole left by $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_1$ and $\mathbf{b}_2$. All this is obvious from the construction if one disregards the four corners of $\mathbf{p}$. (Otherwise see, e.g., [2], for the geometric meaning of a Bézier representation).

More generaly, there is

THEOREM 4.1. *Let* $A = [\mathbf{a}_{i,j}]_{i,j=0}^3$ *and* $B = [\mathbf{b}_{i,j}]_{i,j=0}^3$ *be arbitrary matrices whose entries are points in* R^m, *for some* $m \geqslant 2$, *such that* $\mathbf{a}_{i,j} = \mathbf{b}_{i,j}$, *for* $(i, j) \in \{0, 3\}^2$. *Then* $\mathbf{p}$ *as generated by* blend(A, B) *is continuous and continuously differentiable on* $[0, 1]^2$.

The continuity of $\mathbf{p}$ depends on a parametrization. The simple straightforward parametrization of $\mathbf{p}$ is made explicit in the next section—a more complicated one is needed later. $[0, 1]^2$ is the parameter domain of $\mathbf{p}$ and the fact that $\mathbf{p}$ is continuously differentiable in $[0, 1]^2 \setminus \{0, 1\}^2$ is straightforward.

Theorem (4.1), then, states that $\mathbf{p}$ is also continuously differentiable at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ and $(1, 1)$. The proof is established by a sequence of special results in Sections 6, 7, and 8. The "binder" of these special results is Lemma 5.3.

5. THE SIMPLE PARAMETRIZATION

The simple parametrization of $\mathbf{p}$ is suggested by the standard parametrization of its cubic subpatches. For the sake of clarity this simple parametrization is explicitly defined below.

During its first execution procedure blend generates the matrices A_i and B_i, $i = 1, ..., 4$ (provided $A \neq B$). Then again (if $A_i \neq B_i$) blend (A_i, B_i) generates four matrices A_i and B_i. Let these newly generated matrices be denoted by A_{ij} and B_{ij}, $j = 1, ..., 4$.

More general, let $A_{i_1 \cdots i_n j}$ and $B_{i_1 \cdots i_n j}$ denote the matrices generated at the nth recursion level, i.e., during the execution of blend $(A_{i_1 \cdots i_n}, B_{i_1 \cdots i_n})$.

Analogously we define the points

$$\mathbf{q}_1 = (0, 0), \qquad \mathbf{q}_2 = (0, \tfrac{1}{2}), \qquad \mathbf{q}_3 = (\tfrac{1}{2}, 0), \qquad \mathbf{q}_4 = (\tfrac{1}{2}, \tfrac{1}{2}),$$
$$\mathbf{q}_{i_1 \cdots i_n j} = \mathbf{q}_{i_1 \cdots i_n} + 2^{-n}\mathbf{q}_j \tag{5.1}$$

which correspond to a uniform subdivision of the square $[0, 1)^2$.

For all $\mathbf{i} = i_1 \cdots i_n \in N$, with $A_{\mathbf{i}} = B_{\mathbf{i}}$ the parametrization of $\mathbf{p}$ over $\mathbf{q}_{\mathbf{i}} + [0, 2^{-n}]^2$ is then defined by

$$\mathbf{p}(\mathbf{q}_{\mathbf{i}} + 2^{-n}(u, v)) = \mathscr{B}[A_{\mathbf{i}}](u, v). \tag{5.2}$$

This definition fixes the parametrization of $\mathbf{p}$ besides, maybe, at its corners. There we define—or, perhaps, already have by (5.2)—

$$\mathbf{p}(0, 0) = \mathbf{a}_{0,0} = \mathbf{b}_{0,0}, \qquad \mathbf{p}(0, 1) = \mathbf{a}_{0,3} = \mathbf{b}_{0,3},$$
$$\mathbf{p}(1, 0) = \mathbf{a}_{3,0} = \mathbf{b}_{3,0}, \qquad \mathbf{p}(1, 1) = \mathbf{a}_{3,3} = \mathbf{b}_{3,3}.$$

This definition for, e.g., $\mathbf{p}(0, 0)$ coincides with (5.2) if $A_1 = B_1$. Let $\mathbf{p}[A, B](\cdot)$ denote the simple parametrization of the surface generated by blend (A, B). On exploiting the linearity of the subdivision operator one gets

LEMMA 5.3. *If A, B, C and D are 4×4 matrices over R^m such that $\mathbf{a}_{i,j} = \mathbf{b}_{i,j}$ and $\mathbf{c}_{i,j} = \mathbf{d}_{i,j}$ for all $(i, j) \in \{0, 3\}^2$, then*

$$\mathbf{p}[A + C, B + D](u, v) = \mathbf{p}[A, B](u, v) + \mathbf{p}[C, D](u, v).$$

6. C^1-CONTINUITY WITH THE SIMPLE PARAMETRIZATION

Let $E_{i,j}(\mathbf{d}) = [\mathbf{e}_{k,l}]_{k,l=0}^3$ denote the 4×4 elementary matrix over R^m all of whose entries are zero vectors except $\mathbf{e}_{i,j} = \mathbf{d}$. Then a special case of Theorem (4.1) is covered by

PROPOSITION 6.1. *If $B - A = E_{1,1}(\mathbf{d})$, $\mathbf{d} \in R^m$, then $\mathbf{p}(u, u) \in C^1([0, 1]^2, R^m)$ where $\mathbf{p}(u, v)$ denotes the simple parametrization of the surface generated by* blend(A, B).

Because of symmetry reasons, Proposition (6.1) holds also if $B - A = E_{1,2}(\mathbf{d})$, $B - A = E_{2,1}(\mathbf{d})$, and $B - A = E_{2,2}(\mathbf{d})$ and because of Lemma (5.3) also if $B - A = \sum_{i,j=1}^2 E_{i,j}(\mathbf{d}_{i,j})$.

In order to facilitate the following proofs we introduce the notation

$$M \otimes N := [m_{i,j} n_{i,j}] \tag{6.2}$$

for any two $m \times n$ matrices $M = [m_{i,j}]$ and $N = [n_{i,j}]$.

Proof. If $\mathbf{d} = \mathbf{0}$, then $\mathbf{p}(u, v) = \mathbf{a}(u, v) = \mathbf{b}(u, v)$. Hence, let us assume the nontrivial case, $\mathbf{d} \neq \mathbf{0}$. Let $\bar{n}$ denote the concatenation of n ones, $1 \cdots 1$, i.e., e.g., $A = A_0$. Then procedure blend generates only the matrices $A_{\bar{n}i}$ and $B_{\bar{n}i}$,

$n=0, 1, 2, \ldots,\ i=1, 2, 3, 4$. From the definition of $A_{\tilde n}$ and $B_{\tilde n}$ and the subdivision construction one can derive

$$B_{\tilde n}-A_{\tilde n}=4^{-n}E_{1,1}(\mathbf{d}). \tag{6.3}$$

(6.3) helps to analyse the matrices

$$D_n:=\begin{bmatrix} A_{\tilde n 1} & A_{\tilde n 2} \\ A_{\tilde n 3} & A_{\tilde n 4} \end{bmatrix}-\tilde A_{\tilde n} \tag{6.4}$$

which can also be obtained through the operation

$$D_n=(\tilde B_{\tilde n}-\tilde A_{\tilde n})\otimes P, \tag{6.5}$$

where

$$P=\begin{bmatrix} \bigcirc & I & I & \bigcirc \\ \bigcirc & \Gamma & \Gamma & \bigcirc \\ \bigcirc & \Gamma & \Gamma & \bigcirc \\ \bigcirc & I & I & \bigcirc \end{bmatrix}$$

$$\bigcirc=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad I=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \Gamma=\tfrac{1}{2}I.$$

On introducing the abbreviation

$$\mathbf{d}_n(u, v):=\tilde{\mathscr{B}}[D_n](u, v) \tag{6.6}$$

one derives from (6.4)

$$\mathbf{p}(u, v)=\mathbf{a}(u, v)+\sum_{n=0}^{\infty}\mathbf{d}_n(2^n u, 2^n v) \tag{6.7}$$

and concludes from (6.3) and (6.5) that

$$\max_{[0,1]^2}\|\mathbf{d}_n(u, v)\|\leqslant\max_{i,j}\|D_n[i, j]\|=O(4^{-n}) \tag{6.8}$$

for any vector norm $\|\cdot\|$. Thus, the infinite sum in (6.7) and its partial derivatives converge uniformly, i.e.,

$$\mathbf{p}(u, v)\in C^1[0, 1]^2. \quad \blacksquare$$

7. Incompatible Partial Derivatives

In case the patches **a** and **b** have a common tangent plane in $(0, 0)$ but do not satisfy condition (2.3) in $(0, 0)$, the vectors

$$\mathbf{u} = \mathbf{b}_{1,0} - \mathbf{a}_{1,0} \qquad \text{and} \qquad \mathbf{v} = \mathbf{b}_{0,1} - \mathbf{a}_{0,1}$$

are not both zero. However, $\mathbf{b}_{0,0}$, $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,0}$, $\mathbf{a}_{0,1}$, $\mathbf{a}_{1,0}$ are coplanar. Obviously, the simple parametrization of blend (A, B) cannot be in $C^1(0, 0)$ in this case. Yet, one still has

Proposition 7.1. *Let $B - A = E_{1,0}(\mathbf{u}) + E_{0,1}(\mathbf{v}) \neq (0, 0)$. Then the simple parametrization $\mathbf{p}(u, v)$ of* blend(A, B) *is continuous.*

The remark preceding the proof of (6.1) applies here, too, after adapting it to the context here.

Proof. The proof is similar to the proof of (6.1). Here, (6.3) takes on the form

$$B_n - A_n = \frac{1}{2^n}\left(E_{1,0}(\mathbf{u}) + E_{0,1}(\mathbf{v}) + \left(1 - \frac{1}{2^n}\right) E_{1,1}(\mathbf{u} + \mathbf{v})\right). \tag{7.2}$$

Again, let D_n and $\mathbf{d}_n$ be as in (6.4), (6.5), and (6.6). Then (6.7) is still valid but we only have

$$\max_{[0,1]^2} \|\mathbf{d}_n(u, v)\| = O(2^{-n})$$

for every norm $\|\cdot\|$. This implies, at least, uniform convergence in (6.7) and, hence, continuity of $\mathbf{p}(u, v)$. ∎

While the partial derivatives of $\mathbf{p}(u, v)$ do not exist in $(0, 0)$ they do exist everywhere else in $[0, 1]^2$ and the following two lemmas show that they are bounded. These facts are needed to construct a C^1 parametrization for blend(A, B).

Lemma 7.3. *Under the assumptions of* (7.1) *and the simplification $A = 0$ one gets in a neighborhood of* $(0, 0)$

$$\mathbf{p}_v(u, v) = \mu\mathbf{u} + \nu\mathbf{v}, \qquad (u, v) \neq (0, 0),$$

where $\mu = \mu(u, v) \in (-0.9, 2.5)$ and $\nu = \nu(u, v) \in (-1, 5.2)$.

Proof. From (6.7) and (7.2) we obtain

$$\begin{aligned}\mathbf{p}(u, v) &= \sum_{n=0}^{\infty} \mathbf{d}_n(2^n u, 2^n v)\\ &= q(u, v)\mathbf{u} + r(u, v)\mathbf{v}\end{aligned}$$

where, e.g.,

$$q(u, v) + \sum_{n=0}^{\infty} \frac{1}{2^n} [f + h - 2^{-n} h](2^n u, 2^n v) \tag{7.4}$$

with the piecewise bicubic patches

$$f(u, v) := \mathscr{B}[\tilde{E}_{1,0}(1) \otimes P](u, v)$$

and

$$h(u, v) := \mathscr{B}[\tilde{E}_{1,1}(1) \otimes P](u, v).$$

In order to estimate $\mu = q_v$ in $[0, 1]^2 \setminus \{(0, 0)\}$ we first show that

$$\mu(u, v) = \sum_{n=0}^{\infty} [f_v + h_v](2^n u, 2^n v) + O(u \log u). \tag{7.5}$$

Namely, there is a positive constant M such that

$$|h_v(u, v)| \leqslant \begin{cases} Mu & \text{if} \quad u \in (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $u = 0$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h_v(2^n u, 2^n v) = 0$$

and for $u = 2^{-N} x$, $x \in (\frac{1}{2}, 1]$

$$\begin{aligned}\left| \sum_{n=0}^{\infty} \frac{1}{2^n} h_v(2^n u, 2^n v) \right| &\leqslant \sum_{n=0}^{N} \frac{1}{2^n} M 2^n u\\ &= M(N + 1)u\\ &= M(1 + \log x - \log u)u\\ &= O(u \log u).\end{aligned}$$

This implies (7.5) in view of (7.4). It remains to find a bound for

$$\sum_{n=0}^{\infty} [f_v + h_v](2^n u, 2^n v).$$

After some elementary manipulations one obtains in $[0, 1]^2 \setminus \{(0, 0)\}$

$$-3F(u) \leqslant [f_v + h_v](u, v) \leqslant F(u)$$

where

$$F(u) := \frac{27}{8} \begin{cases} 2u - 6u^2 + 5u^3 & \text{for} \quad u \in [0, \tfrac{1}{2}] \\ u - 2u^2 + u^3 & \text{for} \quad u \in (\tfrac{1}{2}, 1]. \end{cases}$$

If N is such that $x = 2^N u \in (\frac{1}{2}, 1]$, then

$$\sum_{n=0}^{\infty} (f_v + h_v)(2^n u, 2^n v) \leqslant \sum_{n=0}^{N} F(2^n u) = \sum_{n=0}^{N} F(2^{-n} x),$$

i.e.,

$$\sum_{n=0}^{\infty} (f_v + h_v)(2^n u, 2^n v) \leqslant \sum_{n=0}^{\infty} F(2^{-n} x) = \tfrac{27}{8}(3x - 4x^2 + \tfrac{12}{7} x^3)$$
$$< 2.5 \qquad \text{for} \quad x \in [\tfrac{1}{2}, 1].$$

This, essentially, establishes the bounds for μ. Similarly one can construct the bounds for v. ▌

Symmetrically to (7.3) there is

LEMMA 7.6. *Under the same assumptions as in* (7.3) *one gets in a neighbourhood of* (0, 0)

$$\mathbf{p}_u(u, v) = \xi \mathbf{u} + \eta \mathbf{v}, \qquad (u, v) \neq (0, 0),$$

where $\xi \in (-2.2, 4)$ *and* $\eta \in (-2.5, 0.9)$.

The bounds given in (7.3) and (7.6) are rather crude and can be improved.

8. A C^1-PARAMETRIZATION IN THE GENERAL CASE

Let A and B be two 4×4 matrices over R^m, $m \geqslant 2$, such that

$$\mathbf{a}(u, v) = \mathscr{B}[A](u, v)$$

and

$$\mathbf{b}(u, v) = \mathscr{B}[B](u, v)$$

satisfy (2.1) and have common tangent planes in each corner (i, j) of $(0, 1]^2$. Under some constraints on

$$\frac{\partial}{\partial w}(\mathbf{b}-\mathbf{a})(i, j), \qquad w = u, v, \quad (i, j) \in \{0, 1\}^2,$$

we show that blend (A, B) is tangent plane continuous.

Without loss of generality we could assume that A and B are already generated by the first call of blend. Therefore, we can assume that

$$B - A = D + E_{1,1}(\mathbf{w}),$$

where $D = E_{1,0}(\mathbf{u}) + E_{0,1}(\mathbf{v})$. Because of symmetry reasons incompatibilities in other corners can be dealt with analogously.

Let $\mathbf{p}(u, v)$ and $\mathbf{q}(u, v)$ be the simple parametrizations of blend$(A, B-D)$ and respectively blend$(0, D)$. Then recall from (5.3) that $\mathbf{p}+\mathbf{q}$ represents the simple parametrization of blend(A, B) and from (6.1) that $\mathbf{p} \in C^1[0, 1]^2$.

THEOREM 8.1. *Let* $\mathbf{p}_u := \mathbf{p}_u(0, 0) = 3(\mathbf{a}_{1,0} - \mathbf{a}_{0,0})$ *and* $\mathbf{p}_v := \mathbf{p}_v(0, 0) = 3(\mathbf{a}_{0,1} - \mathbf{a}_{0,0})$. *Assume that* $\mathbf{p}_u + \mu\mathbf{u} + \nu\mathbf{v}$ *and* $\mathbf{p}_v + \xi\mathbf{u} + \eta\mathbf{v}$ *are linearly independent for all* μ, ν, ξ, η *in the bounds of* (7.3) *and* (7.6). *Then* blend(A, B) *is tangent plane continuous.*

Proof. In particular, the vectors $\mathbf{p}_u$ and $\mathbf{p}_v$ are linearly independent. Thus there is a linear map $\varphi: R^2 \to \operatorname{span}\{\mathbf{p}_u, \mathbf{p}_v\}$ such that

$$\varphi\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{p}_u \qquad \text{and} \qquad \varphi\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{p}_v.$$

Since image of $\mathbf{q} \subseteq \operatorname{span}\{\mathbf{u}, \mathbf{v}\} \subseteq \operatorname{span}\{\mathbf{p}_u, \mathbf{p}_v\}$, $\varphi^{-1} \circ \mathbf{q}$ maps into R^2. Let

$$k(u, v) := id(u, v) + \varphi^{-1} \circ \mathbf{q}(u, v) \colon R^2 \to R^2.$$

$\mathbf{q}$ and therefore k are differentiable in $[0, 1]\setminus(0, 0)$. Moreover, from (7.3) and (7.6) it follows that the Jacobi matrix of $\varphi \circ k$ satisfies

$$J_{\varphi \circ k} = [\mathbf{p}_u + \xi\mathbf{u} + \eta\mathbf{v}, \mathbf{p}_v + \mu\mathbf{u} + \nu\mathbf{v}]$$

in a neighborhood of $(0, 0)$, where ξ, η, μ, and ν lie in the bounds given in (7.3) and (7.6). Because of the assumption in (8.1), J_k is bounded and the Jacobian of k is bounded away from zero. Thus k is locally invertible in a neighborhood W of $(0,0)$ without $(0,0)$ and $J_{k^{-1}}$ is bounded in W. One can conclude then that the Jacobi matrix of

$$\begin{aligned}(\mathbf{p}+\mathbf{q})\circ k^{-1} &= \mathbf{p}k^{-1}+\mathbf{q}k^{-1}\\ &= \mathbf{p}k^{-1}+\varphi(k-id)k^{-1}\\ &= (\mathbf{p}-\varphi)k^{-1}+\varphi\end{aligned}$$

converges to $J_\varphi(u,v)=J_\mathbf{p}(u,v)$ as (u,v) goes to $(0,0)$. ∎

9. Quick Computation

The recursive procedure blend in Section 3 can easily be transformed into an iterative procedure. A very inexpensive form of the algorithm can be obtained if $B-A=\sum_{i,j=1,2} E_{i,j}(\mathbf{v}_{i,j})$.

First, this quick algorithm evaluates blend $(0, E_{1,1}(1))$ at the grid points $\{0, 1/n, 2/n, ..., n/n\}^2$ using (6.7), where $n=2^k$ is assumed:

0	$S[i,j] := \mathscr{B}[E_{1,1}(1)](i/n, j/n)$,	$i,j=0,...,2^k$
1	Add $4^{-1}S[2i, 2j]$ to $S[i,j]$,	$i,j=0,...,2^{k-1}$
2	Add $4^{-2}S[2^2 i, 2^2 j]$ to $S[i,j]$,	$i,j=0,...,2^{k-2}$
⋮	⋮	
k	Add $4^{-k}S[2^k i, 2^k j]$ to $S[i,j]$,	$i,j=0,1$.

Because of symmetry reasons and Lemma (5.3) one finally has

k + 1 blend(A, B) at $(i/n, j/n)$

$$\begin{aligned}&= \mathbf{a}(i/n, j/n)+S[i,j]\mathbf{v}_{1,1}+S[i,n-j]\mathbf{v}_{1,2}\\ &\quad+S[n-i,j]\mathbf{v}_{2,1}+S[n-j,n-i]\mathbf{v}_{2,2}\end{aligned}$$

where $\mathbf{a}(u,v)=\mathscr{B}[A]$ is the simple parametrization of blend (A, A).

Remark 9.1. The matrix S as computed in the algorithm above can be stored and used again to compute blend(A, B) with different values of A and B. If only l steps of the algorithm are performed, $l \leqslant k$, one obtains an approxmation S_l for the exact value S_k of S. Because of (6.3), (6.5), and (6.7)

$$(S_l-S)[i,j]=\frac{1}{4}(S_{l-1}-S)[2i,2j]=\frac{1}{4^l}(S_0-S)[2^l i, 2^l j].$$

Also, each point blend$(A, B)(u, v)$ lies between $\mathscr{B}[A](u, v)$ and $\mathscr{B}[B](u, v)$. Thus $S_0[i, j]$ and $S[i, j]$ both lie between 0 and $\mathscr{B}[E_{1,1}(1)](i/n, j/n)$, i.e., $|S_0[i, j] - S[i, j]| \leqslant \mathscr{B}[E_{1,1}(1)](u, v) < 0.2$.

Remark 9.2. The derivatives of blend(A, B) can be computed by essentially the same algorithm. For example one gets the u-partial derivative after the following modifications of the above algorithm: Replace $\mathscr{B}[E_{1,1}(1)]$ by $(\partial/\partial u)\mathscr{B}[E_{1,1}(1)]$ in step **0**, the number 4 by 2 in steps **1** through **k**, and in step **k + 1** replace blend(A, B) by $(\partial/\partial u)$ blend(A, B) and $\mathbf{a}$ by $(\partial/\partial u)\mathbf{a}$.

10. Extensions

The procedure blend can be modified and used with polynomials of higher degrees [6]. Then one can obtain also patches of higher smoothness. In particular, let $\mathbf{a}$ and $\mathbf{b}$ in (2.6) and (2.7) be polynomials of degree (m, m), $m > 3$. Then their Bézier matrices A and B are $(m+1) \times (m+1)$ matrices. In order to accommodate blend to these matrices the pattern on the right sides of (3.1) and (3.2) needs to be redefined. There are various possibilities. The pattern matrices in (3.1) are always of the form

$$\begin{bmatrix} P & PE \\ EP & EPE \end{bmatrix}, \qquad \text{where} \quad E = [\delta_{i,m-j}]_{i,j=0}^{m} = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$$

with some $(m+1) \times (m+1)$ pattern matrix P. The corresponding pattern in (3.2) is obtained by transposing the pattern of (3.1) and changing $+$ signs to $-$ signs and vice versa.

Example 10.1. If $m = 4$,

$$P = \begin{bmatrix} + & + & - & - & - \\ + & + & - & - & - \\ + & + & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ \end{bmatrix}$$

and $A[i, j] = B[i, j]$ for $(i, j) \in \{0, 1, 3, 4\}^2 \setminus \{(1, 1), (1, 3), (3, 1), (3, 3)\}$, then blend$(A, B)$ is a C^1 patch which interpolates $\mathbf{a}(i, v)$, $\mathbf{a}_u(i, v)$, $\mathbf{b}(u, i)$, and $\mathbf{b}_v(u, i)$, $i = 0, 1$, $u, v \in [0, 1]$. The proof is analogous to the proof of (6.1).

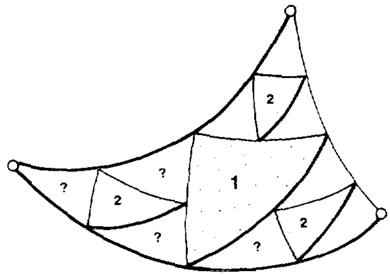

FIG. 3. A triangular patch produced by blending.

EXAMPLE 10.2. If $m = 5$

$$P = \begin{bmatrix} + & + & + & - & - & - \\ + & + & + & - & - & - \\ + & + & + & - & - & - \\ + & + & + & \circ & \circ & \circ \\ + & + & + & \circ & \circ & \circ \\ + & + & + & \circ & \circ & \circ \end{bmatrix}$$

and $A[i, j] = B[i, j]$ for $i + j \leqslant 2$, $i + j \geqslant 8$, $i - j \geqslant 3$, and $j - i \geqslant 3$, then blend(A, B) is a C^2 patch which interpolates $\mathbf{a}(i, v)$, $\mathbf{a}_u(i, v)$, $\mathbf{a}_{uu}(i, v)$, $\mathbf{b}(u, i)$, $\mathbf{b}_v(u, i)$, and $\mathbf{b}_{vv}(u, i)$, $i = 0, 1$, for all $u, v \in [0, 1]$. The proof is analogous to the proof of (6.1).

Another application of blend is obtained if the procedure is formulated for triangular Bézier patches. Instead of blending two quadrangular Bézier patches **a** and **b** the modified procedure has to blend three triangular Bézier patches to a smooth interpolating patch.

Figure 3 illustrates what this modification of blend renders after two recursions. The numbers indicate at which recursion level a subpatch is rendered. More details are given in [7].

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